The equilibrium range in the spectrum of wind-generated waves

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(*Received* 1 *February* 1958)

SUMMARY

Consideration of the structure of wind-generated waves when the duration and fetch of the wind are large suggests that the smaller-scale components of the wave field may be in a condition of statistical equilibrium determined by the requirements for attachment of the crests of the waves. A dimensional analysis, based upon the idea of an equilibrium range in the wave spectrum, shows that for large values of the frequency ω, the spectrum $\Phi(\omega)$ is of the form

$$\Phi(\omega) \sim \alpha g^2 \omega^{-5},$$

where α is an absolute constant. The instantaneous spatial spectrum $\Psi(\mathbf{k})$ is proportional to k^{-4} for large wave numbers k, which is consistent with the observed occurrence of sharp crests in a well-developed sea, and the loss of energy from the wave system to turbulence and heat is proportional to $\rho_w u_*^3$, where ρ_w is the water density and u_* the friction velocity of the wind at the surface. This prediction of the form of $\Phi(\omega)$ for large ω with $\alpha = 7{\cdot}4 \times 10^{-3}$, agrees well with measurements made by Burling (1955).

1. THE CONCEPT OF AN EQUILIBRIUM RANGE

When a turbulent wind commences to blow across a large sheet of water initially at rest, a wave system is initiated and develops under the continued action of the wind. According to a theory described in a recent paper (Phillips 1957), a considerable part in this process is played by a resonance between the convected pressure fluctuations on the surface arising from the turbulence in the wind and the modes of free surface wave propagation, resulting in a rapid growth of the wave amplitudes. This theory, being concerned with the early stages of wave growth, is based upon a linearized surface boundary condition which neglects terms whose magnitudes relative to those retained are of the order of the wave slope. It is predicted that, after a short initial stage, the root-mean-square amplitude of each Fourier component of the wave system increases with time t as $t^{1/2}$. Now, if the turbulent wind continues to blow for a sufficiently long time and if the fetch is sufficiently large, the wave amplitudes continue to increase, so that ultimately the linearized approximation of this theory becomes inadequate and interactions among the Fourier components of the wave field become increasingly important. This present note is concerned with situations in which the fetch and duration of the wind are such that the amplitude of at least some components of the wave field can no longer be considered as 'infinitesimal' and interactions among these components are appreciable.

Some valuable qualitative information on the properties of the wave system can be inferred from even casual observations of wind-generated waves on the ocean. In the first place, the surface displacement at a given point is a random function of time, or at a given instant is a random function of position in the sense that the detailed configuration of the surface is not reproduced by apparently identical meteorological and other conditions. This situation is similar to that found in the study of turbulent motion and suggests that similar statistical concepts should be applied. In particular, only the average properties of the waves can be regarded as significant experimentally or predictable by any reasonable theory. Secondly, it is commonly observed that in a well-developed sea, or one in which there are waves of 'finite height', a characteristic property is the occurrence of fairly sharp wave crests with intermittent patches of foaming ('white-caps' or 'white horses') which seem to develop when the crests can no longer maintain their attachment to the remainder of the water.

The exact nature of this detachment has been considered by many people, and perhaps the most pertinent investigation is that of G. I. Taylor (1953) who studied the limiting configuration of standing waves in a wave tank. In the open ocean, in deep water, it is likely that the limiting configuration of part of the water surface is reached when the local downward acceleration of the fluid near the wave crest is equal to g, the acceleration due to gravity. If the surface configuration changes in such a way that the local acceleration tends to increase beyond this value, that is, in such a way as to put the fluid near the crest in a state of tension, (as happens if the wave crests tend to become sharper), the surface breaks and the crest of the wave detaches. The formation of these sharp crests leading to detachment is observed when two waves of the physical wave pattern run together, or when a wave, travelling with its phase velocity, moves into a region where the energy density of the wave field is locally high. A typical sequence of events in time can be summarized by saying that a wave which is initially rounded, develops a sharp crest which may either subside or detach and disappear, dissolving amidst a foaming patch of turbulence and leaving the wave again rounded. An alternative way of visualizing the same process is to consider an instantaneous photograph of a large region of the sea surface. Such a photograph would show a large number of rounded wave maxima where the local surface acceleration was less than the gravitational acceleration g, and also a smaller number of sharp wave crests where the surface acceleration happened to be exactly equal to g, so that at these crests the surface was on the point of breaking. It would also show some white foam patches marking the demise of sharp wave crests that had reached the limiting state at previous instants and broken.

We now enquire whether these observations can be translated into mathematical language concerning the instantaneous spectrum of the surface displacement. The occurrence of scattered sharp wave crests as a transient limiting configuration corresponds to the occurrence of discontinuities of surface gradient which in spectral terms corresponds to the existence of a certain form of the spectrum of high wave-number. The properties of the instantaneous spatial spectrum at these high wave-numbers will therefore be determined by the physical parameters that determine the extreme configuration of the surface in the limiting condition, the particular property that is relevant being the magnitude of the discontinuity in surface slope developed. It seems likely, therefore, that in a well-developed sea there exists a range of large wave-numbers over which the wave spectrum is statistically determined by the physical quantities governing the conditions for attachment of the wave crests. Similar remarks can be made concerning the frequency spectrum of the surface displacement at a given point, where rapid changes in the surface displacement are associated with the movement past the point of observation of occasional sharp wave crests near the limiting configuration.

The basic hypothesis of this paper can therefore be stated briefly as follows. In a well-developed sea, generated by the wind, there is an 'equilibrium range' of large wave-numbers (or high frequencies) in the spectrum, determined by the physical parameters that govern the continuity of the wave surface.

There are two remarks that should be made at this point. The first is that the concept of an equilibrium range is suggested by consideration of the *asymptotic* form of the spectrum for large wave-numbers (or frequencies) and that we have little direct evidence upon which to base an *a priori* estimate of the smallest wave-number to which we would expect it to be applicable. It is clear that a necessary condition for the existence of an equilibrium range over a certain part of the spectrum is the existence of appreciable non-linear interactions among these wave-numbers, but it is not clear whether this condition alone is sufficient. The results of some measurements described in § 3 offer good *a posteriori* evidence that it may indeed be sufficient but it may be difficult to justify such an assertion in advance. The second remark is that the magnitude of the wave spectrum in the equilibrium range represents an upper limit, dictated by the requirements of crest attachment. In the early stages of wave generation, the equilibrium value may not have been attained at any point in the spectrum, although the wave slopes may be such that non-linear interactions are not negligible, and in a decaying wave system the wave spectrum over the relevant range may have fallen from its equilibrium value through the damping of the components of shorter wavelength. Fruitful consideration of these situations is difficult, even on simple dimensional grounds, because of the additional parameters involved, and we will restrict our attention to the equilibrium range, which might be expected to be attained when a statistically steady wind of sufficient duration and fetch continues to act upon the water surface and supply energy to the wave motion.

2. Dimensional analysis

Since we have defined the equilibrium range of the wave spectrum to be that part governed by the requirements of attachment to the wave crests, it would be expected to be independent of the fetch and duration of the generating wind. The physical quantities to be considered in a dimensional analysis, then, are the densities ρ_a and ρ_w of the air and the water, the friction velocity u_* representative of the wind speed, the surface roughness length z_0, the gravitational acceleration g, the surface tension and viscosity of the water T and ν, together with the wave-number k or the frequency ω. But not all of these parameters will be important. Firstly, provided we restrict attention to wave-numbers and frequencies well below those associated with capillary ripples, so that

$$k \ll \left\{\frac{\rho_w g}{T}\right\}^{1/2}, \qquad \omega \ll \left\{\frac{4\rho_w g^3}{T}\right\}^{1/4}, \tag{1}$$

the influence of surface tension is unimportant. Furthermore, the direct effect of viscosity is manifest in wave damping, the logarithmic decrement being $\gamma = 2\nu k^2$ (Lamb 1932, §§ 348, 349), and, under the conditions with which we are concerned, the rate of energy transfer from the wind to the waves is such that this damping can reasonably be ignored. The condition (1), in the present situation, suffices to ensure that the influence of viscosity in the water is also unimportant. Thirdly, as soon as we agree that T can be omitted from consideration, it will be observed that the dimension mass occurs only in ρ_a and ρ_w, so that if we are to be concerned with quantities that do not contain this dimension, then the densities can occur only as functions of the combination ρ_a/ρ_w. Under oceanographic conditions, with an air–water interface, this ratio is virtually constant, although under more general conditions, at the interface between two fluids, it may vary between zero and unity and its variation may demand consideration.

Of the remaining parameters mentioned above, there is some evidence that the surface roughness length z_0 is not independent of the others. One might reasonably expect the surface roughness to depend upon the wind speed, specified by u_*, and Ellison (1956) suggests that

$$z_0 \propto u_*^2/g. \tag{2}$$

Accurate measurements of z_0 over the sea are difficult to make and the results of different workers are by no means consistent. However, Ellison quotes the results of some careful measurements by Hay (1956) which support a relation of this type and further evidence has been collected by Charnock (1955). The length z_0 is determined by the height of the short steep waves on the water surface and is probably of the order of one-thirtieth of the height of these waves (judging from experience with solid roughness elements), so that Hay's measurements, made for values of z_0 less than 0·3 cm, probably to refer to conditions under which the amplitude of such waves is less than about 10 cm. However, if the expression (2) is valid for these relatively short finite amplitude waves, it is likely to be true for larger values of z_0 and u_*, and we can reasonably conclude that the equilibrium range properties of the wave field are determined only by the parameters u_* and g (and in the general case ρ_a/ρ_w), together with k or ω, provided the wave-numbers and frequencies satisfy the conditions (1).

The functional form of the wave spectrum in the equilibrium range can now be found readily. The frequency spectrum of the surface displacement $\xi(\mathbf{x}, t)$ at a fixed point is defined as

$$\Phi(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} \overline{\xi(\mathbf{x}, t)\xi(\mathbf{x}, t+\tau)} e^{-i\omega\tau}\, d\tau, \tag{3}$$

and the instantaneous wave-number spectrum as

$$\Psi(\mathbf{k}) = (2\pi)^{-2} \int \overline{\xi(\mathbf{x}, t)\xi(\mathbf{x}+\mathbf{r}, t)} e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}, \tag{4}$$

the integration being over the whole surface. The function $\Phi(\omega)$ has dimensions L^2T and $\Psi(\mathbf{k})$ has dimensions L^4. As we have seen, the form of these expressions for large values of their arguments is determined by the limiting configurations attained by the surface under the requirements of attachment of the wave crests. The geometry of the limiting shape near the sharp crests is determined* by the condition that the downward acceleration should not exceed g, so that the asymptotic forms of (3) and (4) would not be expected to involve u_*. An increase in u_* would have the effect of increasing the *rate* at which wave crests are passing through the transient limiting condition of an abrupt change in surface gradient but should not influence the geometry of such a sharp crest itself.

As far as the functions (3) and (4) are concerned, then, the relevant physical parameter seems to be simply g, as well as either ω or k as the case may be, so that it follows immediately on dimensional grounds that

$$\Phi(\omega) \sim \alpha g^2 \omega^{-5}, \tag{5}$$

where α is a constant, and

$$\Psi(\mathbf{k}) \sim f(\theta) k^{-4}, \tag{6}$$

where θ is an angle specifying the direction of the vector wave-number $\mathbf{k}$ and $f(\theta)$ is determined by the directional distribution of the smaller-scale components. These expressions might be expected to refer to ranges of frequency and wave-number such that

$$\omega_0 \ll \omega \ll (4\rho_w g^3 T^{-1})^{1/4}; \qquad k_0 \ll k \ll (\rho_w g T^{-1})^{1/2}, \tag{7}$$

where ω_0 and k_0 represent the smallest frequency and wave-number of the wave field for which non-linear effects are important.

It is interesting to notice that the expression (6) is consistent with the particular form that the limiting configuration assumes, that is, with the occurrence of sharp wave crests. It can be shown that if a function has simple discontinuities in gradient at one or more points, then its Fourier transform is asymptotically proportional to k^{-2}. Thus the wave spectrum, which is essentially the mean-square of the Fourier transform, is proportional

* We here exclude a different possible type of surface instability in which the sharp crests may be ' blown off ' by very high winds.

to k^{-4} for large k. Indeed, this remark suggests that (6) can be obtained alternatively by replacing our argument that u_* is not involved in the equilibrium range of the spectrum at any instant by the specific premise that the limiting configuration is characterized by the occurrence of sharp wave crests. Then, it follows from the statements immediately above that $\Psi(\mathbf{k}) \propto k^{-4}$, with a coefficient of proportionality that may now involve u_* as well as g. But this coefficient must be dimensionless, and since there is no dimensionless combination of u_* and g, the coefficient must be constant for each direction of the vector wave-number $\mathbf{k}$. This argument is in some ways more attractive than the previous one leading to (6), though the specific nature of the premise used precludes an immediate deduction of the result (5) for $\Phi(\omega)$.

Another important property of the wave field that is determined by the equilibrium range of the spectrum is ϵ, the mean rate at which energy is lost from the wave motion on unit area of surface either to turbulence by wave breaking or by direct dissipation in the smaller scale components. The process of formation and detachment of sharp wave crests corresponds, though possibly not in a simple way, to energy transfer across the equilibrium range to high frequency components with its subsequent loss from the wave motion. If the range of frequencies from which energy is extracted from the wind is distinct from the range over which energy is lost, then ϵ represents the rate at which energy is transferred across the equilibrium range of the wave spectrum, and so is determined by u_* and g alone. The dimensions of ϵ are (energy)/(time × area), so that

$$\epsilon = \beta \rho_w u_*^3 \tag{8}$$

where β is a dimensionless constant. If the internal energy dissipation in the waves is neglected, ϵ represents the difference between the rate of gain of energy from the wind and the rate of increase of energy of those components of the wave system that are still developing and have not yet attained a statistical equilibrium state.

3. Measurements of the high frequency components of the wave spectrum

Among the most reliable measurements yet available of the wave spectrum $\Phi(\omega)$ at large frequencies are those of Burling (1955), made on Staines Reservoir, Middlesex, England, using the capacitance wire recorder developed by Tucker & Charnock (1955). This instrument records accurately the high frequency components of the wave system and so is particularly suited to an investigation of the existence and properties of any equilibrium range in the wave spectrum. A considerable number of wave records were made over fetches of from 400 m to 1350 m, with wind velocities at a height of 10 m ranging from 500 to 800 cm/sec. The results of taking Fourier analyses of these records are shown in figure 1. When the frequency ω is small, the spectrum curves depend quite strongly on the fetch and meteorological conditions, but their most remarkable property is that when ω is large, the curves obtained under different conditions very nearly coincide and become apparently independent of the fetch and of the strength of the wind. So striking was this observation that Burling himself suggested the possibility of some type of equilibrium structure, but the idea was not followed up. He found empirically that the mean values of his observed spectra in this range obeyed a relation of the type

$$\Phi(\omega) = 7{\cdot}0 \times 10^3 \omega^{-5}$$

in c.g.s. units, where $\Phi(\omega)$ is as defined in (3) and the frequency ω is expressed in radians per second. This frequency dependence is the same as has been predicted by (5) and the observed value of the coefficient indicates that the dimensionless constant α in (5) is approximately $7{\cdot}4 \times 10^{-3}$. These measurements appear to offer good support both to the original hypothesis of the existence of an equilibrium range and to the result (5) based on this idea.

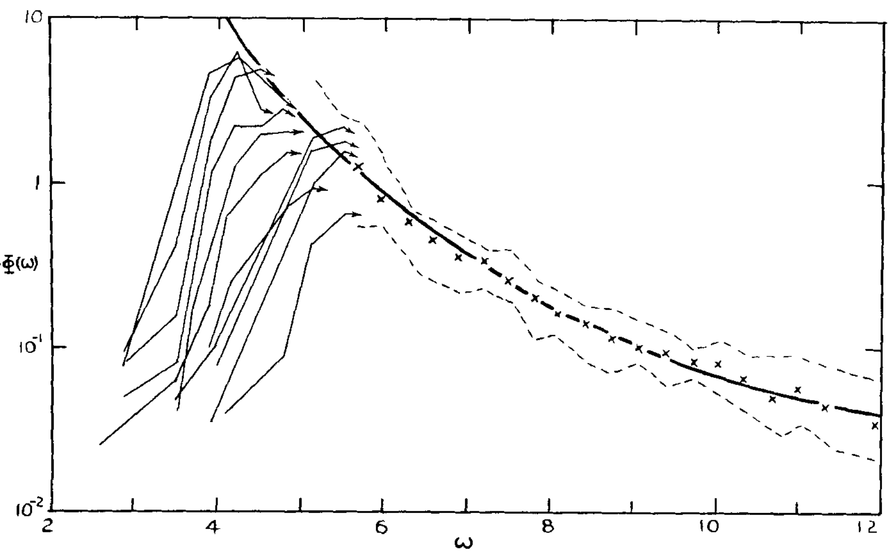

Figure 1. Spectra of wind-generated waves measured by Burling (1955). The cluster of lines on the left are representative of the spectra at low frequencies for which equilibrium has not been attained. On the right the curves merge over the equilibrium range, and the broken lines indicate the extreme measured values of $\Phi(\omega)$ at each frequency ω. The crosses represent the mean observed value at each ω, and the heavy line the relation $\Phi(\omega) = \alpha g^2 \omega^{-5}$ with $\alpha = 7{\cdot}4 \times 10^{-3}$.

It is interesting to compare these results with those obtained by other oceanographers which are summarized in various empirical or semi-empirical spectra proposed as descriptions of the sea surface. Bretschneider (1958) proposes a spectrum of the form

$$s(\omega) = \alpha g^2 \omega^{-5} \exp\{-0{\cdot}675(2\pi/T\omega)^4\},$$

where T is a 'mean wave period' which depends upon wind speed and the state of development of the sea. When ω is large, Bretschneider's spectrum, which is dimensionally sound, reduces to the form (5) derived above, and Bretschneider finds that the value $\alpha = 7{\cdot}4 \times 10^{-3}$ obtained from Burling's measurements is consistent with his own observations. On the other hand, Neumann (1954) proposed the expression

$$I(\omega) = C\omega^{-6} \exp\{-2g^2/\omega^2 W^2\},$$

for a 'fully developed sea', where W is the 'wind speed' and C is a dimensional constant equal to $1{\cdot}40 \times 10^4\,\mathrm{cm\,sec^{-3}}$, $I(\omega)$ being normalized in the same way as our definition (3). The presence of this dimensional constant C cannot easily be justified on dimensional grounds, since it must be determined by some physical parameters yet is independent of ω, W (or u_*) and fetch, presumably leaving only g which cannot account for the dimensions of C. If we put this important objection aside, however, and examine Neumann's spectrum as it stands, it is clear that when ω is large

$$I(\omega) \sim C\omega^{-6}, \tag{9}$$

which is independent of the wind speed and so contains one of the properties of the equilibrium range as described in this note. But the frequency dependence is as ω^{-6} rather than as ω^{-5}, which is required by dimensional analysis and supported by Burling's observations; and the latter deserve more weight in view of the more adequate and precise instrumentation. Furthermore, the magnitude of $I(\omega)$ predicted by (9) is only about 25% of the values observed by Burling in this frequency range. However, in fairness to the Neumann spectrum as an empirical formula for describing ocean waves, it should be mentioned that the data from which Neumann obtained his spectrum cover frequencies much lower than those observed by Burling, so that (9) represents a considerable extrapolation from frequency ranges over which this spectrum has been found to give a reasonable empirical fit. Since it seems that the Neumann spectrum is unlikely to represent a genuine physical law on account of its dimensional inconsistency, its failure at these high frequencies is hardly surprising, and of course does not damage its usefulness in practical wave forecasting.

This work was supported by Contract Nonr 248(38) with the Mechanics Branch of the Office of Naval Research.

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